FILTERED FROBENIUS ALGEBRAS IN MONOIDAL CATEGORIES

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joint work Chelsea Walton)

① Motivation

(i) From noncommutative ring theory

Filtered algebras	Graded algebras
$A_0 \subset A_1 \subset A_2 \cdots$	$B = \oplus B_i$
$A = \bigcup_{i \in \mathbb{N}_0} A_i$	$B_i \cdot B_j \subset B_{i+j}$
$A_i \cdot A_j \subset A_{i+j}$	

- $A \xmapsto{\ gr\ } gr(A) = \bigoplus_{i \in \mathbb{N}} \frac{A_i}{A_{i-1}}$ (associated graded algebra of A)
- A is called a filtered deformation of gr(A).

$\rightarrow$ gr(A) is a graded algebra

Example:

A	gr(A)
$\mathcal{U}(\mathfrak{g})$	$S(\mathfrak{g})$
$Cl(V, B)$	$\Lambda(V)$

In fact, many nice properties of gr(A) transfer to A.

If gr(A) is integral domain / Noetherian / prime then so is gr(A).

The talk is about the property of being Frobenius.

But, $\boxed{\text{what}}$ are Frobenius algebras and $\boxed{\text{why}}$ should we care about them?

This brings us to the second motivation

(ii) From Quantum algebra

$$\underbrace{\text{Frobenius algebras}}\ \text{in}\ \underbrace{\text{monoidal category}}_{=(\mathcal{C},\otimes,\mathbb{1})}$$

- (like vector spaces) $\mathcal{C}$ = a category
- (like tensor of vector spaces) $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ bifunctor
- (like ground field $\Bbbk$) $\mathbb{1}$: unit object of $\mathcal{C}$ unit for $\otimes$ product

$\Downarrow$

(allows us to do algebra in $\mathcal{C}$)

WHAT?

A Frobenius algebra in $(\mathcal{C},\otimes,\mathbb{1})$ is a 5-tuple

algebra in $\mathcal{C}$:
- A = object in $\mathcal{C}$
- $m : A\otimes A \longrightarrow A$
- $u : \mathbb{1} \longrightarrow A$

coalgebra in $\mathcal{C}$:
- A = object in $\mathcal{C}$
- $\Delta : A \longrightarrow A\otimes A$
- $\varepsilon : A \longrightarrow \mathbb{1}$

+

$$\begin{tikzcd}
 & A\otimes A \arrow[dl, "\Delta\otimes id"'] \arrow[d, "m"] \arrow[dr, "id\otimes\Delta"] & \\
A\otimes A \arrow[dr, "id\otimes m"'] & A \arrow[d, "\Delta"] & A\otimes A\otimes A \arrow[dl, "m\otimes id"] \\
 & A\otimes A &
\end{tikzcd}$$

<u>Example</u>: $\Bbbk G$ with $\Delta(g) = \sum_{h\in G} gh^{-1}\otimes h$

$\varepsilon(g) = \delta_{g,e_G}$

is a Frobenius algebra in $\mathcal{C} = \mathrm{Vec}_{\Bbbk}$.

WHY?

Frobenius algebras in monoidal categories, show up in work on

(i) TQFTs and CFTs

(ii) Morita theory

(iii) Classification of subfactors

(iv) Computer Science

With these motivations in mind,
the following NC ring theory result provides a context for our work.

Theorem (Bongale, 1967)

Let A be a finite dimensional connected $(A_0 = \Bbbk)$, filtered algebra over $\Bbbk$. If $gr(A)$ is Frobenius, then so is A.

$\downarrow$ we generalize this to get our main result

MAIN THEOREM (Walton-Y., 21)

Let $\mathcal{C}$ be an abelian, rigid monoidal category. Let A be a connected, filtered algebra in $\mathcal{C}$ with finite filtration. If $gr(A)$ is a Frobenius algebra in $\mathcal{C}$, then so is A.

As an application of this, we are able to prove that.

Theorem (Walton-Y.)

Every exact module category $\mathcal{M}$ over a symmetric finite tensor category $\mathcal{C}$ is represented by a Frobenius algebra A in $\mathcal{C}$, i.e., $\mathcal{M} = \mathcal{C}_A$.

(more details at the end)

Let's come back to proving the main theorem.
We need to develop two tools to prove it.

(i) Associated graded algebra construction
(ii) New Characterization of Frobenius algebras.

(i) Associated graded functor

For $\mathcal{C}$ an abelian, monoidal category with $\otimes$ biexact. We construct a monoidal associated graded functor

$$\mathrm{gr}: \mathrm{Fil}(\mathcal{C}) \longrightarrow \mathrm{Gr}(\mathcal{C})$$

$$(A, F_A) \longmapsto \coprod_{i \in \mathbb{N}_0} \frac{F_A(i)}{F_A(i-1)}$$

Objects: (X, F_X) with $X \in \mathcal{C}$, $F_X : \underline{\mathbb{N}_0} \to \mathcal{C}$ s.t. $\mathrm{colim}_i F_X(i) = X$

Morphisms: $X \xrightarrow{f} Y$ in $\mathcal{C}$ that preserve filtration

$(X, F_X) \otimes (Y, F_Y) := \left(X \otimes Y, \begin{smallmatrix}\text{Day convolution} \\ \text{of } F_X, F_Y\end{smallmatrix}\right)$

Objects: $X = \coprod_{i \in \mathbb{N}_0} X_i$

Morphisms: $X \to Y$ compatible with $\coprod$ decomposition

$(X \otimes Y)_k = \coprod_{i+j=k} (X_i \otimes Y_j)$

$\boxed{\text{UPSHOT}}$ If A is a filtered algebra in $\mathcal{C}$, then $\mathrm{gr}(A)$ is a graded algebra in $\mathcal{C}$.

(ii) New characterization of Frobenius algebras

Theorem
Let $\mathcal{C}$ be a rigid monoidal category. An algebra (A, m, u) in $\mathcal{C}$ is Frobenius

if and only if

$\exists\ \nu : A \to \mathbb{1}$ such that any left/right ideal of A that factors through $\ker(\nu)$ is zero.

Proof sketch of the main theorem:

- (A, F_A) has finite filtration $\Rightarrow$ $A \cong F_A(n)$ for some $n \in \mathbb{N}$

- Take $\nu : A \xrightarrow{\sim} F_A(n) \longrightarrow \dfrac{F_A(n)}{F_A(n)} \cong \mathbb{1}$ because A is connected

- Take any ideal I of A so that

$$\begin{tikzcd} \ker(\nu) \arrow[r] & A \arrow[r, "\nu"] & \mathbb{1} \\ I \arrow[u] \arrow[ur, "\phi"'] & & \end{tikzcd}$$

(by our characterization of Frobenius algs, it suffices to) show that $I = 0$

- Consider $gr(\phi) : gr(I) \to gr(A)$

- Show $gr(\phi)$ factors through the kernel of the Frobenius form on $gr(A)$. Hence,

$$gr(I) = 0$$

- Hence, $I = 0$.

x —— x

Directions for future work

① Generalize and study other ring theoretical properties that lift under filtered deformations

② Construct braided Clifford algebras and show that they are Frobenius by showing that its associated graded is the exterior algebra.

Details of the application

Take $\mathcal{C}$ = symmetric finite tensor category
$\mathcal{M}$ = exact module category over $\mathcal{C}$

Etingof - Ostrik showed that $\mathcal{M} = \mathcal{C}_A$ where

$$A = \mathrm{Ind}_{\hat{H}}^{H}\left(\mathrm{Ind}_{H}^{\hat{H}}(\mathrm{End}(V)) \otimes Cl_W \right)$$

Key idea : $gr(Cl_W) = \Lambda_W$ is the exterior algebra
+
Λ_W is Frobenius
$\Rightarrow$ Cl_W is Frobenius

- Clearly $\mathrm{End}(V)$ is also Frobenius
- Ind_H^G are Frobenius monoidal
- $\otimes$ of Frob. algs is Frobenius

$\Rightarrow$ A is Frobenius.